INVOLUTORIAL TRANSFORMATIONS IN S_3 OF ORDER n WITH AN $(n-1)$-FOLD LINE*

BY

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1. **Introduction.** Montesano† has given a brief synthetic discussion of the existence of involutorial transformations I_n of order n with an $(n-1)$-fold line l. He showed that the planes through l are interchanged in pairs by I_n and that the lines in one plane are transformed into lines in the conjugate plane. He also showed that the I_n could be defined by the aid of two curves of order $n-1$ situated on the fundamental surface $F_{n-1}:l^{n-2}$ which is the image of l.

In this paper the $F_{n-1}:l^{n-2}$ and an $F_n:l^{n-1}$ are used to define an involutorial transformation of order $2n-1$ with a $(2n-3)$-fold line which, if certain conditions are satisfied, reduces to an involutorial transformation of order n with an $(n-1)$-fold line. The explicit analytical forms of I_3 and I_4 are found by this method. For larger values of n it is convenient to define I_n by other means. There is a net of surfaces of order m, $m+1$, or $m+2$ according as $n=3m-1$, $3m$, or $3m+1$, which is transformed by I_n into a net of surfaces of order m. These nets are used to define the involutorial transformation and the equations of I_5, I_6, and I_7 are derived. A method is given for mapping I_n on ordinary space so that it is apparent that I_n is rational.

2. **The birational transformation of type (n, n) with an $(n-1)$-fold line.** Two surfaces F_n of order n having an $(n-1)$-fold line l in common, meet in a residual curve C_{2n-1}. Any plane through l meets each F_n in a residual line, and the two lines meet in a point on the C_{2n-1}. Hence the C_{2n-1} meets l in $2n-2$ points. If the two surfaces have a C_n in common meeting l in $n-1$ points, then by Noether's‡ formulas the C_{2n-1} will consist of the C_n and $n-1$ lines l_i meeting l.§ Conversely through $n-1$ lines l_i meeting l, pass ∞^{n+2} surfaces $F^n:l^{n-1}$, Σl_i such that any two meet in a C_n which meets the line l in $n-1$ points and each line l_i in one point. Three of these surfaces meet in n points and if we fix $n-1$ of the points, we have a homoloidal web of surfaces $F_n:l^{n-1}$, Σl_i, ΣP_i.

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† D. Montesano, *Su una classe di trasformazioni involutorie dello spazio*, Istituto Lombardo, Rendiconti, (2), vol. 21 (1888), pp. 688–690.

‡ M. Noether, *Sulle curve multiple di superficie algebriche*, Annali di Matematica, (2), vol. 5 (1871), pp. 163–177.

§ H. P. Hudson, *Cremona Transformations in Plane and Space*, p. 316.

There exists therefore a Cremona transformation $T_{n,n}$ between two spaces (x) and (x') by which a plane in (x') $\sim$ a surface of the web of F_n in (x). A plane and a surface F_n in the (x) space $\sim$ a surface F_n' and a plane in the (x') space. The curves of intersection correspond and have the same genus.* Hence a plane section of F_n' is a C_n' with an $(n-1)$-fold point and the surfaces F_n' must have a common $(n-1)$-fold line l'. Two planes in (x) $\sim$ two F_n' in (x') meeting in a residual C_{2n-1}' which meets l' in $2n-2$ points. The line of intersection of the two planes $\sim$ a non-composite C_n' which can meet l' in not more than $n-1$ points. Therefore the residual C_{n-1}' must meet l' in $n-1$ points, i.e. C_{n-1}' consists of $n-1$ lines l_i'. The C_n' meets l' in exactly $n-1$ points and each l_i' in one point. Three such surfaces F_n' meet in n points of which $n-1$ must be fixed in order to have a homoloidal web.

Among the surfaces of the web in the (x) space there is a pencil consisting of a plane through l and the fixed $F_{n-1}:l^{n-2}$, Σl_i, ΣP_i. Therefore there is a pencil of planes through l' in (x') which corresponds to this pencil of the web of F_n, and l' $\sim$ the fixed F_{n-1}.

A general line in (x') $\sim$ a $C_n:\Sigma P_i$, hence P_i $\sim$ a plane. Since C_n meets each l_i once, l_i $\sim$ a plane. The plane σ_i through l and P_i $\sim$ a plane ρ_i' through l', but since P_i $\sim$ a plane, $P_i \sim \rho_i'$, and the plane σ_i apart from P_i and l $\sim$ a curve s' in ρ_i'. A general plane of (x') meets ρ_i' in a line L' and the curve s' in one or more points Q'. The corresponding F_n meets σ_i in one line L through P_i. Therefore the curve s' is a line. The line L', except for the point Q', $\sim P_i$ and $Q' \sim L$. The line s' must be a fundamental line l_i' because the points of $s' \sim$lines in σ_i through P_i. In a similar manner it is seen that $l_i \sim$a plane σ_i' through l' and P_i'. The plane ρ_i through l and $l_i \sim$the point P_i' in σ_i'. Therefore the points and lines P_i and l_i are associated in pairs with the lines and points l_i' and P_i' respectively.

3. **The involutorial transformation.** When the two spaces are superimposed for the involutorial case, the fundamental systems must coincide and the planes through l are interchanged in pairs by the involutorial transformation I_n. If $x_1=0$ and $x_2=0$ are the invariant planes of this pencil, the four equations of I_n can be obtained from three homogeneous equations of the following form:

$$(1_1) \qquad x_1' = x_1,$$

$$(1_2) \qquad x_2' = -x_2,$$

$$(1_3) \qquad x_3' = (d + ex_3 + fx_4)/(a + bx_3 + cx_4),$$

* G. Loria, *Sulla classificazione delle trasformazioni di genere zero*, Istituto Lombardo, Rendiconti, (2), vol. 23 (1890), pp. 824–834.

where $a=a(x_1, x_2)$, etc., $(d+ex_3+fx_4)=0$ is any F_n of the web, and $(a+bx_3+c_4)=0$ is the fixed F_{n-1}. Since we are dealing with involutorial transformations the inverse of equations (1) have the same form as (1). If in the inverse of (1_3) we replace x_1', x_2' by $x_1, -x_2$ we have

$$x_3 = (\bar{d} + \bar{e}x_3' + \bar{f}x_4')/(\bar{a} + \bar{b}x_3' + \bar{c}x_4'), \tag{2}$$

where $\bar{a}=a(x_1, -x_2)$, etc. This equation can be solved for x_4' and thus we get the fourth equation of the involutorial transformation as

$$\begin{aligned} x_4' = \{ [(\bar{e}d + \bar{d}a) - x_3(\bar{b}d + \bar{a}a)] + x_3[(\bar{e}e + \bar{d}b) - x_3(\bar{b}e + \bar{a}b)] \\ + x_4[(\bar{e}f + \bar{d}c) - x_3(\bar{b}f + \bar{a}c)]\}/[(a + bx_3 + cx_4)(\bar{c}x_3 - \bar{f})]. \end{aligned} \tag{3}$$

When the conditions that $\bar{c}x_3-\bar{f}$ be a factor of the numerator are satisfied and this factor is removed, we have the I_n with an $(n-1)$-fold line, defined analytically.

4. **The cubic case.** A non-homogeneous coördinate system is useful in the cases when $n=3$ or 4, so we put $x_2/x_1=\lambda$, $x_3/x_1=x$, and $x_4/x_1=y$. When $n=3$ there are only two fundamental points P_1, P_2; any plane through the line joining them is transformed by I_3 into another such plane and the two planes $\rho_i=0$, which are the planes l, l_i. Among the planes of the pencil on the line P_1P_2 there are at least two which are invariant. Let $x=0$ and $y=0$ be two of the invariant planes, and let $\rho_i\equiv\lambda_i-\lambda=0$. The points P_1, P_2 are then determined by the planes $\sigma_i\equiv\lambda_i+\lambda=0$ and the line $x=y=0$. One surface of the web is $\rho_1\rho_2x=0$, and the equation of the fixed quadric determined by l, $2l_i$, $2P_i$ is of the form

$$\sigma_1\sigma_2 + (a_0 + a_1\lambda)x + (b_0 + b_1\lambda)y = 0.$$

We can write the first two equations of I_3 as follows:

$$\lambda' = -\lambda, \tag{4_1}$$

$$x' = h\rho_1\rho_2x/\{\sigma_1\sigma_2 + (a_0 + a_1\lambda)x + (b_0 + b_1\lambda)y\}. \tag{4_2}$$

If we write the inverse of (4_2) and replace λ' by $-\lambda$ we have

$$x = h\sigma_1\sigma_2x'/\{\rho_1\rho_2 + (a_0 - a_1\lambda)x' + (b_0 - b_1\lambda)y'\}. \tag{5}$$

When (5) is solved for y', it has the form

$$\begin{aligned} y' = \rho_1\rho_2\{(h^2 - 1)\sigma_1\sigma_2 - x[a_0(h + 2) - a_1(h - 1)\lambda] \\ - y(b_0 + b_1\lambda)\}/[\{\sigma_1\sigma_2 + (a_0 + a_1\lambda)x + (b_0 + b_1\lambda)y\}(b_0 - b_1\lambda)]. \end{aligned} \tag{6}$$

Since $y=0$ is an invariant plane the first two terms in the numerator of (6) must vanish, and the coefficient of y must be divisible by $b-b_1\lambda$. This requires that

$$b_1 = 0, \quad h = \pm 1, \quad a_0(h+1) - a_1(h-1)\lambda = 0.$$

The third condition presents two cases, namely

$$h = 1, \qquad \text{so that } a_0 = 0,$$

or

$$h = -1, \text{ so that } a_1 = 0.$$

The cubic involutorial transformation may now be written in the form

$$\begin{aligned} \lambda' &= -\lambda, \\ x' &= x\rho_1\rho_2/(\sigma_1\sigma_2 + a_1\lambda x + b_0 y), \\ y' &= -y\rho_1\rho_2/(\sigma_1\sigma_2 + a_1\lambda x + b_0 y), \end{aligned} \tag{7}$$

or

$$\begin{aligned} \lambda' &= -\lambda, \\ x' &= -x\rho_1\rho_2/(\sigma_1\sigma_2 + a_0 x + b_0 y), \\ y' &= -y\rho_1\rho_2/(\sigma_1\sigma_2 + a_0 x + b_0 y). \end{aligned} \tag{8}$$

In the first case when $h=1$, the pencil of planes through P_1, P_2 is invariant. In the second case when $h=-1$, each plane of the pencil is invariant.

5. **The quartic case.** There are three fundamental points; one of the surfaces of the web consists of the plane $P_1P_2P_3$ and of the three planes l, l_i. We can take the points P_1, P_2 as in I_3 and the plane $P_1P_2P_3$ as $x=0$. The lines l_i in the planes $\rho_i \equiv \lambda_i - \lambda = 0$ and the points P_i lie in the planes $\sigma_i \equiv \lambda_i + \lambda = 0$. The equation of the fixed $F_3: l^2, 3l_i, 3P_i$ may be written in the form

$$\sigma_1\sigma_2(1-\lambda) + (a_0 + a_1\lambda + a_2\lambda^2)x + (b_0 + b_1\lambda + b_2\lambda^2)y = 0.$$

The first two equations of the I_4 are now given by

$$\lambda' = -\lambda, \tag{9$_1$}$$

$$x' = h\rho_1\rho_2\rho_3 x/[\sigma_1\sigma_2(1-\lambda) + (a_0 + a_1\lambda + a_2\lambda^2)x + (b_0 + b_1\lambda + b_2\lambda^2)y]. \tag{9$_2$}$$

The inverse of (9_2) with λ' replaced by $-\lambda$ is

$$x = h\sigma_1\sigma_2\sigma_3 x'/[\rho_1\rho_2(1+\lambda) + (a_0 - a_1\lambda + a_2\lambda^2)x' + (b_0 - b_1\lambda + b_2\lambda^2)y']. \tag{10}$$

If (10) is solved for y' we have

$$\begin{aligned} y' = \rho_1\rho_2\{&\sigma_1\sigma_2(h^2\rho_3\sigma_3 - 1 + \lambda^2) - x[h\rho_3(a_0 - a_1\lambda + a_2\lambda^2) \\ &+ (1+\lambda)(a_0 + a_1\lambda + a_2\lambda^2)] - y(1+\lambda)(b_0 + b_1\lambda + b_2\lambda^2)\}/\{[b_0 \\ &- b_1\lambda + b_2\lambda^2][\sigma_1\sigma_2(1-\lambda) + (a_0 + a_1\lambda + a_2\lambda^2)x + (b_0 + b_1\lambda + b_2\lambda^2)y]\}. \end{aligned} \tag{11}$$

The expressions

(12) $$\sigma_1\sigma_2(h^2\rho_3\sigma_3 - 1 + \lambda^2),$$

(13) $$h\rho_3(a_0 - a_1\lambda + a_2\lambda^2) + (1 + \lambda)(a_0 + a_1\lambda + a_2\lambda^2),$$

(14) $$(1 + \lambda)(b_0 + b_1\lambda + b_2\lambda^2)$$

must therefore contain the factor $b_0 - b_1\lambda + b_2\lambda^2$. From (14) we find that $b_1 = 0$ and if we use this in (12) we have the condition

(15) $$(1 - h^2)/(\lambda_3^2 h^2 - 1) = b_2/b_0.$$

From (13) we get the conditions

(16) $$[a_2(1 + h\lambda_3) + a_1(1 + h)]/[a_0(1 + h\lambda_3)] = b_2/b_0,$$

(17) $$(1 - h)a_2/[a_1(1 - h\lambda_3) + a_0(1 - h)] = b_2/b_0.$$

These last two conditions may be rewritten as

(16′) $$(h\lambda_3 + 1)(a_0b_2 - a_2b_0) = a_1b_0(1 + h),$$

(17′) $$(1 - h)(a_0b_2 - a_2b_0) = a_1b_2(h\lambda_3 - 1).$$

If we divide (17′) by (16′) we get condition (15) over again so that (15) is included in (16) and (17). We can solve (16) and (17) for h and obtain

(18) $$h = (a_0b_2 - a_2b_0 + a_1b_2)/(a_0b_2 - a_2b_0 + \lambda_3a_1b_2),$$

(19) $$h = (a_1b_1 - a_0b_2 + a_2b_0)/(\lambda_3a_0b_2 - \lambda_3a_2b_0 - a_1b_0);$$

if we equate these values of h we get

(20) $$(\lambda_3 + 1)(a_0b_2 - a_2b_0)^2 + 2a_1(a_0b_2 - a_2b_0)(b_2\lambda_3 - b_0) - a_1^2b_0b_2(\lambda_3 + 1) = 0.$$

The quartic involutorial transformation is therefore determined by the equations

(21)
$$\lambda = -\lambda,$$
$$x' = \frac{hx\rho_1\rho_2\rho_3}{\sigma_1\sigma_2(1 - \lambda) + (a_0 + a_1\lambda + a_2\lambda^2)x + (b_0 + b_2\lambda^2)y},$$
$$y' = \frac{\rho_1\rho_2\{\sigma_1\sigma_2(1 - h^2) - x[a_2(1 + h\lambda_3) + a_1(1 + h) + a_2\lambda(1 - h) - b_2y(1 + \lambda)\}}{b_2[\sigma_1\sigma_2(1 - \lambda) + (a_0 + a_1\lambda + a_2\lambda^2)x + (b_0 + b_2\lambda^2)y]}.$$

In these equations h is defined by (18) or (19) and the coefficients a_i and b_i are subject to condition (20).

6. **The quintic case.** There is a net of quadrics through l and the $4P_i$ which is invariant under the I_5. We can use the vertices of the tetrahedron of reference for the $4P_i$ and take

$$X_1 \equiv d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0,$$
$$X_2 \equiv x_1 + x_2 + x_3 + x_4 = 0$$

as the invariant planes through l, so that

$$(22)\qquad \begin{aligned} x_1' &= X_1F_4, \\ x_2' &= -X_2F_4, \end{aligned}$$

where F_4 is determined by l^3, $4l_i$, $4P_i$. The planes l, P_i are given by $\sigma_i \equiv X_1 - d_iX_2 = 0$ and the planes l, l_i by $\rho_i \equiv X_1 + d_iX_2 = 0$. The net of quadrics has the form

$$k_1\sigma_1x_1 + k_2\sigma_2x_2 + k_3\sigma_3x_3 = 0$$

and from (22) we have the identity

$$\sigma_1x_1 + \sigma_2x_2 + \sigma_3x_3 + \sigma_4x_4 = 0.$$

The quadrics of the net are interchanged in pairs involutorially by I_5, so that the involutorial transformation can be defined by

$$(23)\qquad \begin{aligned} (a_1\sigma_1'x_1' + a_2\sigma_2'x_2' + a_3\sigma_3'x_3') &= (a_1\sigma_1x_1 + a_2\sigma_2x_2 + a_3\sigma_3x_3)\rho_2\rho_3\rho_4F_4, \\ (b_1\sigma_1'x_1' + b_2\sigma_2'x_2' + b_3\sigma_3'x_3') &= (b_1\sigma_1x_1 + b_2\sigma_2x_2 + b_3\sigma_3x_3)\rho_1\rho_3\rho_4F_4, \\ (c_1\sigma_1'x_1' + c_2\sigma_2'x_2' + c_3\sigma_3'x_3') &= -(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)\rho_1\rho_2\rho_4F_4, \end{aligned}$$

and the identity

$$(24)\qquad \sigma_1'x_1' + \sigma_2'x_2' + \sigma_3'x_3' + \sigma_4'x_4' = 0.$$

If we solve (23) for x_i' replacing σ_i' by ρ_i and use (24) to obtain x_4', we have the I_5 expressed by

$$(25)\qquad \begin{aligned} x_1' &= [\sigma_1x_1\Delta - 2C_1(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)]\rho_2\rho_3\rho_4, \\ x_2' &= [\sigma_2x_2\Delta - 2C_2(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)]\rho_1\rho_3\rho_4, \\ x_3' &= [\sigma_3x_3\Delta - 2C_3(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)]\rho_1\rho_2\rho_4, \\ x_4' &= [\sigma_4x_4\Delta + 2(C_1 + C_2 + C_3)(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)]\rho_1\rho_2\rho_3, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

and C_i is the cofactor of c_i in Δ.

7. **The involutorial transformations I_6, I_7, and I_n.** There is a net of $F_3{:}l^2, 5P_i, l_5$ which is transformed into a net of $F_2{:}l, 4P_i (i<5)$ by I_6. Among the cubics of the net there is the pencil of $F_2{:}l, 5P_i$ with the fixed component

$\rho_5=0$, which is invariant under I_6. Hence using the same coordinate system as in the quintic case we can determine the I_6 by

$$\begin{aligned}
a_1\sigma_1' x_1' + a_2\sigma_2' x_2' + a_3\sigma_3' x_3' &= (a_1\sigma_1x_1 + a_2\sigma_2x_2 + a_3\sigma_3x_3)\rho_1\rho_2\rho_3\rho_4\rho_5F_5,\\
b_1\sigma_1' x_1' + b_2\sigma_2' x_2' + b_3\sigma_3' x_3' &= -(b_1\sigma_1x_1 + b_2\sigma_2x_2 + b_3\sigma_3x_3)\rho_1\rho_2\rho_3\rho_4\rho_5F_5,\\
c_1\sigma_1' x_1' + c_2\sigma_2' x_2' + c_3\sigma_3' x_3' &= F_3\rho_1\rho_2\rho_3\rho_4F_5,\\
d_1x_1' + d_2x_2' + d_3x_3' + d_4x_4' &= (d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4)F_5,\\
x_1' + x_2' + x_3' + x_4' &= -(x_1 + x_2 + x_3 + x_4)F_5,
\end{aligned}$$

where F_5 is the fixed quintic surface. The a_i and b_i are restricted since these quadrics must contain P_5. The cubic F_3 is of the form

$$\begin{aligned}
F_3 \equiv (g_1X_1^2 + g_2X_1X_2 + g_3X_2^2)(g_4x_1 + g_5x_2 + g_6x_3)&\\
+ \rho_5(g_7\sigma_1x_1 + g_8\sigma_2x_2 + g_9\sigma_3x_3) &= 0,
\end{aligned}$$

where $g_4x_1+g_5x_2+g_6x_3=0$ is the plane through l_5, P_4; and $g_7\sigma_1x_1+g_8\sigma_2x_2+g_9\sigma_3x_3=0$ is a quadric of the pencil l, $5P_i$. The a_i, b_i, c_i, d_i, g_i must satisfy the conditions necessary in order that F_3 may be transformed by I_6 into

$$(c_1\sigma_1x_1 + c_2\sigma_2x_2 + c_3\sigma_3x_3)\rho_1\rho_2\rho_3\rho_4\rho_5\sigma_5F_5^2 .$$

When $n=7$ there is a net of quartic surfaces $F_4:l^3$, $6P_i$, l_5, l_6 which correspond to a net of quadrics $F_2:l$, $4P_i(i<5)$. Among the surfaces of the net there is a pencil of $F_3:l^2$, $6P_i$, l_5 with $\rho_6=0$ as a fixed component, which is transformed into the pencil of $F_2:l$, $5P_i(i<6)$. Among the surfaces of the pencil there is the cubic consisting of the plane $\rho_5=0$ and the quadric $F_2:l$, $6P_i$ which is invariant under I_7. The equations which determine the I_7 are therefore of the following form:

$$\begin{aligned}
a_1\sigma_1' x_1' + a_2\sigma_2' x_2' + a_3\sigma_3' x_3' &= (a_1\sigma_1x_1 + a_2\sigma_2x_2 + a_3\sigma_3x_3)\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6F_6,\\
b_1\sigma_1' x_1' + b_2\sigma_2' x_2' + b_3\sigma_3' x_3' &= F_3\rho_1\rho_2\rho_3\rho_4\rho_5F_6,\\
c_1\sigma_1' x_1' + c_2\sigma_2' x_2' + c_3\sigma_3' x_3' &= F_4\rho_1\rho_2\rho_3\rho_4F_6,\\
d_1x_1' + d_2x_2' + d_3x_3' + d_4x_4' &= (d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4)F_6,\\
x_1' + x_2' + x_3' + x_4' &= -(x_1 + x_2 + x_3 + x_4)F_6.
\end{aligned}$$

The forms obtained for $n=5, 6, 7$ can be generalized as follows:

A. If $n=3m-1$ there is a net of $F_m:l^{m-1}$, ΣP_i.

B. If $n=3m$ there is a net of $F_{m+1}:l^m$, ΣP_i, l_{3m-1} containing the pencil of $F_m:l^{m-1}$, ΣP_i with a fixed component the plane l, l_{3m-1}.

C. If $n=3m+1$ there is a net of $F_{m+2}:l^{m+1}$, ΣP_i, l_{3m-1}, l_{3m} containing a pencil of $F_{m+1}:l^m$, ΣP_i, l_{3m-1} with the fixed component l, l_{3m}. One of the surfaces of the pencil is the $F_m:l^{m-1}$, ΣP_i with the fixed component l, l_{3m-1}.

In each case the net is transformed into a net of $F_m:l^{m-1}$, $\Sigma P_i(i<3m-1)$ by I_n, hence I_n can be defined by means of the nets.

8. **The mapping of the involutorial transformation I_n.** The expressions $2x_1x_1'$, $-2x_2x_2'$, $x_1x_3'+x_3x_1'$, $x_2x_3'+x_3x_2'$ are invariant under I_n. Let us consider the correspondence between the (x) space and a (y) space where the values of x_i' above are those defined in §3, (1). The correspondence has the form

$$
(26)\quad
\begin{aligned}
y_1 &= 2x_1^2(a+bx_3+cx_4),\\
y_2 &= 2x_2^2(a+bx_3+cx_4),\\
y_3 &= x_1[(d+ex_3+fx_4)+x_3(a+bx_3+cx_4)],\\
y_4 &= x_2[(d+ex_3+fx_4)-x_3(a+bx_3+cx_4)].
\end{aligned}
$$

These equations can be solved for x_i as follows:

$$
(27)\quad
\begin{aligned}
x_2/x_1 &= \pm\,(y_2/y_1)^{1/2},\\
x_3 &= -\,x_1\bar{U}/(x_2y_1),\\
x_4 &= [dx_2^2y_1^2 - ax_1x_2y_1U - ex_1x_2y_1\bar{U} + bx_1^2U\bar{U}]/[cx_1x_2y_1U - fx_2^2y_1^2],
\end{aligned}
$$

where $U=x_1y_4+x_2y_3$, $\bar{U}=x_1y_4-x_2y_3$. Hence equations (26) define a (1, 2) correspondence.

If we rewrite the equations of the correspondence in terms of x_i' and replace x_1', x_2' by x_1, $-x_2$, we have

$$
(28)\quad
\begin{aligned}
y_1 &= 2x_1^2(\bar{a}+\bar{b}x_3'+\bar{c}x_4'),\\
y_2 &= 2x_2^2(\bar{a}+\bar{b}x_3'+\bar{c}x_4'),\\
y_3 &= x_1[(\bar{d}+\bar{e}x_3'+\bar{f}x_4')+x_3'(\bar{a}+\bar{b}x_3'+\bar{c}x_4')],\\
y_4 &= -\,x_2[(\bar{d}+\bar{e}x_3'+\bar{f}x_4')-x_3'(\bar{a}+\bar{b}x_3'+\bar{c}x_4')],
\end{aligned}
$$

where $\bar{a}=a(x_1,-x_2)$, etc. If we equate the values of y_i given in (26) and (28) we have

$$
(29)\quad
\begin{aligned}
&a+bx_3+cx_4=\bar{a}+\bar{b}x_3'+\bar{c}x_4',\\
&(d+ex_3+fx_4)+x_3(a+bx_3+cx_4)\\
&\qquad\qquad=(\bar{d}+\bar{e}x_3'+\bar{f}x_4')+x_3'(\bar{a}+\bar{b}x_3'+\bar{c}x_4'),\\
&(d+ex_3+fx_4)-x_3(a+bx_3+cx_4)\\
&\qquad\qquad=-\,(\bar{d}+\bar{e}x_3'+\bar{f}x_4')+x_3'(\bar{a}+\bar{b}x_3'+\bar{c}x_4').
\end{aligned}
$$

From (29) we get

$$
(30)\quad
\begin{aligned}
x_3' &= (d+ex_3+fx_4)/(a+bx_3+cx_4),\\
x_3 &= (\bar{d}+\bar{e}x_3'+\bar{f}x_4')/(\bar{a}+\bar{b}x_3'+\bar{c}x_4'),
\end{aligned}
$$

but these are precisely the equations §3, (1), (2) by which the involutorial transformation I_n was defined.

Hence we see that a (1, 2) correspondence of the type given by equations (26) leads in general to a special type of involutorial transformation of order $2n-1$ with a $(2n-3)$-fold line. If however conditions are imposed that $\bar{c}x_3-\bar{f}$ be a factor of the numerator of the expression for x_4', then $\bar{c}x_3-\bar{f}$ is a factor of x_1', x_2', x_3', and x_4' and we have an I_n with an $(n-1)$-fold line. We have therefore proved the

THEOREM. *An involutorial transformation in S_3 of order n with an $(n-1)$-fold line is rational.*

9. **Image of a general line in (y).** A line $y_3=Ay_1+By_2$, $y_4=Cy_1+Dy_2$ in the (y) space is transformed by the correspondence into the C_{n+4} given parametrically by the equations

$$
\begin{aligned}
X_1 &= x_1^2x_2(cV-fx_1x_2),\\
X_2 &= x_1x_2^2(cV-fx_1x_2),\\
X_3 &= -\overline{V}(cV-fx_1x_2),\\
X_4 &= dx_1^2x_2^2-x_1x_2(aV+e\overline{V})+bV\overline{V},
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
V &= x_1(Cx_1^2+Dx_2^2)+x_2(Ax_1^2+Bx_2^2),\\
\overline{V} &= x_1(Cx_1^2+Dx_2^2)-x_2(Ax_1^2+Bx_2^2).
\end{aligned}
$$

In the case of the I_3 the curve is a C_7 of the form

$$
\begin{aligned}
X_1 &= b_0x_1^3x_2V,\\
X_2 &= b_0x_1^2x_2^2V,\\
X_3 &= -b_0x_1V\overline{V},\\
X_4 &= x_2[a_1V\overline{V}-x_1(\sigma_1\sigma_2V+\rho_1\rho_2\overline{V})].
\end{aligned}
\tag{32}
$$

If we put $V=0$ in (32), there are three values of the parameter x_2/x_1 all giving the point (0, 0, 0, 1). If we put $\overline{V}=0$, we get three distinct points in the invariant plane $x_3=0$. When $x_1=0$, we again have the point (0, 0, 0, 1) and furthermore the C_7 is tangent to l at that point with $x_1=0$ as the osculating plane. When we put $x_2=0$, we have the point (0, 0, 1, 0). Hence the C_7 has a fourfold point (0, 0, 0, 1) at which it is also tangent to l and passes through the point (0, 0, 1, 0).

In the general case there are $n+1$ values of x_2/x_1 due to the vanishing of $(cV-fx_1x_2)$ which give the point (0, 0, 0, 1). When $x_1=0$ or $x_2=0$, we get two definite points on l at which the C_{n+4} is tangent to the planes $x_1=0$, $x_2=0$ respectively. A plane of the pencil $y_2=k^2y_1$ has for images the two planes $x_2=\pm kx_1$ which meet the C_{n+4} in the two images of the point in which $y_2=k^2y_1$ meets the line of which the C_{n+4} is the image.

10. **Image of a line in** (y) **which meets** l'. Any line in (y) meeting l' may be defined by

$$y_2 = k^2y_1, \quad Ay_1 + By_2 + Cy_3 + Dy_4 = 0.$$

The image in (x) of such a line is a pair of conics each belonging to a net in the planes $x_2=\pm kx_1$. In the plane $x_2=kx_1$ the net has the form

$$2x_1(a_{11}x_1 + b_{11}x_3 + c_{11}x_4)(A + k^2B) + x_1(d_{11}x_1 + e_{11}x_3 + f_{11}x_4)(C + kD) + x_3(a_{11}x_1 + b_{11}x_3 + c_{11}x_4)(C - kD) = 0,$$

where $a_{11}=a(1, k)$, etc. The conics of the net pass through the fixed points

$$x_1 = x_2 = b_{11}x_3 + c_{11}x_4 = 0,$$
$$x_1 = x_2 = x_3 = 0,$$
$$x_2 - kx_1 = a_{11}x_1 + b_{11}x_3 + c_{11}x_4 = d_{11}x_1 + e_{11}x_3 + f_{11}x_4 = 0.$$

Two lines in $y_2=k^2y_1$ have for images a pair of conics of each net; the point of intersection of the two lines corresponds to the two free intersections of the two pairs of conics.

In the case of the invariant plane $y_2=0$, the lines in $y_2=0$ correspond to a pencil of conics in the plane $x_2=0$ given by

$$2Ax_1(a_{10}x_1+b_{10}x_3+c_{10}x_4)+C[x_1(d_{10}x_1+e_{10}x_3+f_{10}x_4)+x_3(a_{10}x_1+b_{10}x_3+c_{10}x_4)]=0$$

where $a_{10}=a(1, 0)$, etc. The pencil of conics has the three fixed points

$$x_1 = x_2 = b_{10}x_3 + c_{10}x_4 = 0,$$
$$x_1 = x_2 = x_3 = 0,$$
$$x_2 = a_{10}x_1 + b_{10}x_3 + c_{10}x_4 = d_{10}x_1 + e_{10}x_3 + f_{10}x_4 = 0.$$

The variable point of intersection of the net of conics is in this case replaced by the direction of the tangent to

$$x_1(d_{10}x_1 + e_{10}x_3 + f_{10}x_4) + x_3(a_{10}x_1 + b_{10}x_3 + c_{10}x_4) = 0$$

at the point $x_1=x_2=b_{10}x_3+c_{10}x_4=0$. Hence this point is an invariant point the image of which in (y) is $y_2=0$. Similarly the plane $y_1=0$ is the image of the invariant point $x_1=x_2=b_{01}x_3+c_{01}x_4=0$. The surface of branch points in the (y) space consists of the two planes $y_1=0$, $y_2=0$, and the corresponding surface of coincidences in (x) reduces to the two invariant points.

Addendum

In a recently published article* Snyder discusses involutorial birational transformations contained multiply in a linear line complex and suggests that they are probably irrational. The transformation he considers is of order $2k$ with a $(2k-1)$-fold line $x_3=x_4=0$, and $2k-1$ fundamental points lying on the line $x_1=x_2=0$, and so is a special case of the involutorial transformations studied in this paper. The equations of the I_{2k} are given as

$$\begin{aligned} x_1' &= (x_3^{2k-1} + x_4^{2k-1})x_1, \\ x_2' &= (x_3^{2k-1} + x_4^{2k-1})x_2, \\ x_3' &= (x_4^{2k-1} - x_3^{2k-1})x_3, \\ x_4' &= (x_3^{2k-1} - x_4^{2k-1})x_4. \end{aligned}$$

This involutorial transformation may be mapped, as in the general case, on ordinary space by the (1, 2) correspondence given by the equations

$$\begin{aligned} y_1 &= x_1x_4x_3^{2k-1}, \\ y_2 &= x_2x_4x_3^{2k-1}, \\ y_3 &= x_3^2(x_3^{2\ -1} - x_4^{2k-1}), \\ y_4 &= x_4^2(x_3^{2\ -1} - x_4^{2k-1}), \end{aligned}$$

and is therefore rational.

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* V. Snyder, *The simplest involutorial transformation contained multiply in a line complex*, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 89–93.

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